

HIGHER DIMENSIONAL KNOTS IN TUBES

BY
YAICHI SHINOHARA⁽¹⁾

Abstract. Let K be an n -knot in the $(n+2)$ -sphere and V a tubular neighborhood of K . Let L' be an n -knot contained in a tubular neighborhood V' of a trivial n -knot and L the image of L' under an orientation preserving diffeomorphism of V' onto V . The purpose of this paper is to show that the higher dimensional Alexander polynomial and the signature of the n -knot L are determined by those of K and L' .

1. Introduction. An n -knot K is a smooth oriented submanifold of the oriented $(n+2)$ -sphere S^{n+2} which is homeomorphic to S^n . Throughout the paper we assume that the orientation of S^{n+2} is fixed. Two n -knots K_1 and K_2 are said to be of the same n -knot type if there exists an orientation preserving homeomorphism f of S^{n+2} onto itself such that $f(K_1) = K_2$ and $f|_{K_1}$ is orientation preserving. By a *tube* (or an *open tube*) of an n -knot K we mean a closed (or an open) tubular neighborhood of K in S^{n+2} .

Let K be an n -knot in S^{n+2} and V a tube of K . Let V' be a tube of a trivial n -knot K' . Let $f: V' \rightarrow V$ be a diffeomorphism of V' onto V which preserves the orientations induced by S^{n+2} in V' and V . If $n=1$, we further assume that f transforms longitudes into longitudes. Let L' be an n -knot contained in the interior of V' . Then $L' \sim \lambda K'$ in V' for some integer λ . Moreover, $L = f(L')$ is an n -knot contained in the interior of V and $L \sim \lambda K$ in V .

For the case $n=1$, H. Seifert [4] showed that

$$\Delta_L(t) = \Delta_K(t^\lambda) \Delta_{L'}(t),$$

where $\Delta_L(t)$, $\Delta_K(t)$ and $\Delta_{L'}(t)$ are the Alexander polynomials of L , K and L' , and the author [5] proved that

$$\begin{aligned} \sigma(L) &= \sigma(L') && \text{when } \lambda \text{ is even,} \\ &= \sigma(K) + \sigma(L') && \text{when } \lambda \text{ is odd,} \end{aligned}$$

where $\sigma(L)$, $\sigma(K)$ and $\sigma(L')$ are the signatures of L , K and L' defined by H. F. Trotter [6].

The purpose of this paper is to generalize these results to the case of higher dimensional knots.

Received by the editors May 18, 1970.

AMS 1969 subject classifications. Primary 5520; Secondary 5720.

Key words and phrases. q th dimensional Alexander polynomial, signature of n -knots.

⁽¹⁾ The author is a postdoctoral fellow of the National Research Council of Canada (Grant A4034).

Copyright © 1971, American Mathematical Society

Following J. W. Milnor [3], we will define the q th dimensional Alexander polynomial $\Delta_K^q(t)$ of an n -knot K (see §2). We prove that

THEOREM 1.1. *For $1 \leq q \leq n$,*

$$\Delta_L^q(t) = \Delta_K^q(t^\lambda) \cdot \Delta_L^q(t).$$

In [2], D. Erle defined the signature σ_2 of a $(2m-1)$ -knot which is equivalent to σ for $m=1$ [1]. For the case when m is even, another signature σ_1 of a $(2m-1)$ -knot was also defined. We show that

THEOREM 1.2. *If $n=2m-1$, then*

$$\begin{aligned} \sigma_2(L) &= \sigma_2(L') && \text{when } \lambda \text{ is even,} \\ &= \sigma_2(L') + (-1)^{(m+1)(\lambda-1)/2} \sigma_2(K) && \text{when } \lambda \text{ is odd.} \end{aligned}$$

Furthermore, if m is even, then $\sigma_1(L) = \sigma_1(L') + \lambda \sigma_1(K)$.

2. Preliminaries. Z and Q will denote the ring of integers and the field of rational numbers respectively. Throughout the paper we assume that every covering space is connected.

Let X be a connected, locally path-connected and semilocally 1-connected space and Y a subspace of X which is connected and locally path-connected. Suppose that there exists a homomorphism

$$\varphi: \pi_1(X, y_0) \rightarrow G$$

of $\pi_1(X, y_0)$ onto an abelian group G , where y_0 is a point of Y . Then there exists a regular covering space $p: \tilde{X} \rightarrow X$ which belongs to $\text{Ker}(\varphi)$. G acts on $\tilde{X}$ as the group of covering transformations. More precisely, one can describe the action of G on $\tilde{X}$ as follows:

Choose a point $\tilde{y}_0 \in p^{-1}(y_0)$. Let g be an element of G and γ an element of $\pi_1(X, y_0)$ which is mapped into g by φ . Then there exists a unique path class $\tilde{\gamma}$ in $\tilde{X}$ with the initial point $\tilde{y}_0$ such that $p_\#(\tilde{\gamma}) = \gamma$. The terminal point of $\tilde{\gamma}$ does not depend on the choice of γ which satisfies $\varphi(\gamma) = g$. Hence g corresponds to a unique covering transformation which maps $\tilde{y}_0$ into the terminal point of $\tilde{\gamma}$. This correspondence is an isomorphism of G onto the group of covering transformations and does not depend on the choice of $\tilde{y}_0 \in p^{-1}(y_0)$ (since G is abelian).

The above described action of G on $\tilde{X}$ is called the φ -action of G on $\tilde{X}$. For every subgroup S of G , the φ -action of G naturally induces an action of S on $\tilde{X}$.

Now we want to consider $p^{-1}(Y)$. Let H be the image of $\pi_1(Y, y_0)$ under the homomorphism

$$\psi = \varphi \circ i_\#: \pi_1(Y, y_0) \rightarrow G,$$

where $i: Y \subset X$. Let $\{g_1 = 1, g_2, \dots, g_\mu\}$ be a coset representative system of $G \bmod H$; that is, $G = H + g_2H + \dots + g_\mu H$, where $\mu = [G:H]$ may not be finite. Let Y_1 be a path-component of $p^{-1}(Y)$ and $Y_i = g_i(Y_1)$ for $i = 1, 2, \dots, \mu$. Then we have the following lemma.

LEMMA 2.1. (1) $p|Y_i: Y_i \rightarrow Y$ is a covering space which belongs to $\text{Ker}(\psi)$, $i=1, 2, \dots, \mu$.

(2) For $i=1, 2, \dots, \mu$, the φ -action of H on $\tilde{X}$ induces an action of H on Y_i by restricting every element of H to Y_i . Furthermore, this induced action of H on Y_i coincides with the ψ -action of H on Y_i .

(3) $p^{-1}(Y)$ is the disjoint union of $Y_1, Y_2, \dots, Y_\mu$.

This lemma can be proved easily by using basic properties of covering spaces. Hence we omit the details.

Let M be a finitely generated module over a principal ideal domain P . By the structure theorem for finitely generated modules over principal ideal domains, M is isomorphic to a direct sum of cyclic modules; that is,

$$M \cong P/(p_1) \oplus \cdots \oplus P/(p_r),$$

where (p_i) denotes the principal ideal spanned by an element $p_i \in P$. The generator of the product ideal $(p_1 \cdots p_r)$, unique up to unit elements of P , will be called the *order* of M and denoted by $\text{order}_P M$ [3]. The order of M is an invariant of the P -isomorphism class of M .

Now, in Lemma 2.1, suppose that X is a finite connected simplicial complex, Y a connected subcomplex of X and $G=(t:)$ the infinite cyclic group generated by t . Then $H=(t^\lambda:)$ for some integer λ and $p: \tilde{X} \rightarrow X$ is an infinite cyclic covering space. Let $\tilde{Y}$ denote $p^{-1}(Y)$. Since $\tilde{Y}$ is invariant under t and Y is a finite complex, $H_*(\tilde{Y}; Q)$ is a finitely generated Γ -module, where Γ denotes the rational group ring of G .

Suppose $\lambda \neq 0$. Then $[G:H]=|\lambda|=\mu$ and $\{1, t, \dots, t^{\mu-1}\}$ forms a coset representative system of $G \bmod H$. Let Y_1 be a path-component of $\tilde{Y}$ and $Y_i=t^{i-1}(Y_1)$ for $i=1, 2, \dots, \mu$. By Lemma 2.1, $p|Y_i: Y_i \rightarrow Y$ is an infinite cyclic covering space having t^λ as a generator of the group of covering transformations. Hence $H_*(Y_i; Q)$ is a finitely generated Γ^λ -module, where Γ^λ is the rational group ring of H . Moreover, as Γ^λ -modules,

$$H_q(\tilde{Y}; Q) \cong H_q(Y_1; Q) \oplus \cdots \oplus H_q(Y_\mu; Q).$$

Since t_* is an automorphism of $H_q(\tilde{Y}; Q)$ which maps $H_q(Y_1; Q)$ onto $H_q(Y_2; Q)$, $\dots$, $H_q(Y_\mu; Q)$ onto $H_q(Y_1; Q)$, a presentation of the Γ^λ -module $H_q(Y_1; Q)$ can be considered as a presentation of the Γ -module $H_q(\tilde{Y}; Q)$. This yields

COROLLARY 2.2. If $\Delta \in \Gamma^\lambda$ is the order of the Γ^λ -module $H_q(Y_1; Q)$, then Δ is the order of the Γ -module $H_q(\tilde{Y}; Q)$.

Suppose $\lambda=0$. Then H is trivial and $\{t^i\}_{i \in \mathbb{Z}}$ is a coset representative system for $G \bmod H$. Let Y_0 be a path-component of $\tilde{Y}$ and $Y_i=t^i(Y_0)$ for every integer i . By Lemma 2.1, $p|Y_i: Y_i \rightarrow Y$ is a homeomorphism and $\tilde{Y}$ is the disjoint union of Y_i , $i \in \mathbb{Z}$. Hence we have

COROLLARY 2.3. *If $\dim_Q H_q(Y; Q) = \rho$, then $H_q(\tilde{Y}; Q)$ is a free Γ -module of rank ρ .*

Let K be an n -knot in S^{n+2} and X the complement of an open tube U of K in S^{n+2} . Let $\varphi: \pi_1(X) \rightarrow G$ be a homomorphism of $\pi_1(X)$ onto an infinite cyclic group $G = \langle t \rangle$ defined by

$$(2.1) \quad \varphi(g) = t^{\text{Link}(g, K)} \quad \text{for } g \in \pi_1(X).$$

Then the covering space $p: \tilde{X} \rightarrow X$ belonging to $\text{Ker}(\varphi)$ is an infinite cyclic covering space. $H_q(\tilde{X})$ is a finitely generated Λ -module, where Λ is the integral group ring of G . Likewise $H_q(\tilde{X}; Q) \cong H_q(\tilde{X}) \otimes_{\mathbb{Z}} Q$ is a finitely generated Γ -module. The order of the Γ -module $H_q(\tilde{X}; Q)$ will be called the q th dimensional Alexander polynomial of K and denoted by $\Delta_K^q(t)$ [3].

It follows from [2, p. 102] that

PROPOSITION 2.4. *The family $\{\Delta_K^q(t)\}_q$ is an invariant of the n -knot type of K .*

PROPOSITION 2.5. (1) $H_q(\tilde{X}) \xrightarrow{t-1} H_q(\tilde{X})$ is a Λ -isomorphism for $q \neq 0$.

(2) $H_q(\tilde{X})(H_q(\tilde{X}; Q))$ is a torsion $\Lambda(\Gamma)$ -module for every integer q .

(3) $H_{n+1}(\tilde{X}) \cong 0$.

(4) *In the homology sequence of the pair $(\tilde{X}, \partial\tilde{X})$, $\partial_*: H_{n+1}(\tilde{X}, \partial\tilde{X}) \rightarrow H_n(\partial\tilde{X})$ is a Λ -isomorphism.*

Proof. The proof of Assertion 5 in [3] holds for the integral homologies provided that $H_*(X) \cong H_*(S^1)$, from which (1) and (2) follow immediately.

Since there exists an $(n+1)$ -dimensional subcomplex of $\tilde{X}$ which is a deformation retract of $\tilde{X}$, $H_{n+1}(\tilde{X})$ is a free Λ -module. Hence, by (2), $H_{n+1}(\tilde{X}) \cong 0$.

By (3), ∂_* is one-one. Hence it remains to show that $H_n(\partial\tilde{X}) \rightarrow H_n(\tilde{X})$ is trivial, but this follows from (1) and the fact $H_n(\partial\tilde{X}) \cong \Lambda/(t-1)$. Q.E.D.

Finally we want to define the signature of a $(2m-1)$ -knot K in S^{2m+1} .

By Proposition 2.5(4), $\partial_*: H_{2m}(\tilde{X}, \partial\tilde{X}) \rightarrow H_{2m-1}(\partial\tilde{X})$ is an isomorphism which is compatible with t_* . On ∂X , we select an oriented $(2m-1)$ -sphere S^{2m-1} which is homologous to K in $\text{Cl}(U)$. If $m=1$, we further require that S^{2m-1} is a longitude of $\text{Cl}(U)$. Let e be the homology class in $H_{2m-1}(\partial\tilde{X})$ represented by a lifting of S^{2m-1} to $\partial\tilde{X}$. Then e generates $H_{2m-1}(\partial\tilde{X})$ and is called the *canonical generator* of $H_{2m-1}(\partial\tilde{X})$. e is uniquely determined by K and satisfies $t_*(e) = e$. The element $\zeta = \partial_*^{-1}(e)$ of $H_{2m}(\tilde{X}, \partial\tilde{X})$ will be called the *fundamental class* of $\tilde{X}$. ζ is uniquely determined by K and satisfies $t_*(\zeta) = \zeta$.

By the universal-coefficient theorem for cohomology, we have

$$H^{2m}(\tilde{X}, \partial\tilde{X}; Q) \cong \text{Hom}(H_{2m}(\tilde{X}, \partial\tilde{X}; Q), Q), \quad u \mapsto \langle u, \zeta \rangle.$$

Hence we can define two types of pairings B_1 and B_2 from $H^m(\tilde{X}, \partial\tilde{X}; Q) \otimes H^m(\tilde{X}, \partial\tilde{X}; Q)$ to Q by

$$B_1(x, y) = \langle x \cup y, \zeta \rangle$$

and

$$B_2(x, y) = \langle x \cup t^*y + y \cup t^*x, \zeta \rangle$$

for $x, y \in H^m(\tilde{X}, \partial\tilde{X}; Q)$. By Lemma 4.4 of [2], we have

PROPOSITION 2.6. B_1 is a dual pairing and $B_1(y, x) = (-1)^m B_1(x, y)$ for

$$x, y \in H^m(\tilde{X}, \partial\tilde{X}; Q).$$

For every positive integer m , B_2 is a quadratic form on the finite-dimensional vector space $H^m(\tilde{X}, \partial\tilde{X}; Q)$. The signature of B_2 will be denoted by $\sigma_2(K)$.

If m is even, B_1 is also a quadratic form on $H^m(\tilde{X}, \partial\tilde{X}; Q)$ and the signature of B_1 will be denoted by $\sigma_1(K)$.

Theorem 5.2 of [2] shows that

PROPOSITION 2.7. $\sigma_1(K)$ and $\sigma_2(K)$ are invariants of the $(2m-1)$ -knot type of K .

3. **Proof of Theorem 1.1.** Let K, V, L and λ be as in Theorem 1.1. Let U be a tube of L which is contained in $\text{Int } V$.

We will use the following notations:

$$\begin{aligned} X &= S^{n+2} - \text{Int } U, & i_1: W &\subset X, \\ W &= V - \text{Int } U, & i_2: Y &\subset X, \\ Y &= S^{n+2} - \text{Int } V, & i_3: T &\subset X, \\ T &= \text{Bdary } V. \end{aligned}$$

Choose a point $x \in T$. Let φ be a homomorphism of $\pi_1(X, x)$ onto an infinite cyclic group $G = (t:)$ defined by

$$\varphi(g) = t^{\text{Link}(g, L)} \quad \text{for } g \in \pi_1(X, x),$$

and $p: \tilde{X} \rightarrow X$ the infinite cyclic covering space belonging to $\text{Ker } (\varphi)$. Put $\tilde{W} = p^{-1}(W)$, $\tilde{Y} = p^{-1}(Y)$ and $\tilde{T} = p^{-1}(T)$. Then $\tilde{X} = \tilde{W} \cup \tilde{Y}$ and $\tilde{T} = \tilde{W} \cap \tilde{Y}$.

First we will consider the case $\lambda \neq 0$. We want to show that if $\lambda \neq 0$ then

$$(3.1) \quad \begin{aligned} \Delta_L^q(t) &= \Delta_K^q(t^\lambda) \cdot \text{order}_\Gamma H_q(\tilde{W}; Q), & 1 \leq q \leq n-1, \\ (t^\lambda - 1) \cdot \Delta_L^n(t) &= \Delta_K^n(t^\lambda) \cdot \text{order}_\Gamma H_n(\tilde{W}; Q). \end{aligned}$$

Since $L \sim \lambda K$ in V , the homomorphism

$$\pi_1(Y, x) \xrightarrow{i_{2\#}} \pi_1(X, x) \xrightarrow{\varphi} G$$

is given by

$$(3.2) \quad (\varphi \cdot i_{2\#})(g) = t^{\lambda \cdot \text{Link}(g, K)}$$

for $g \in \pi_1(Y, x)$, and $\pi_1(Y, x)$ is mapped onto $H = (t^\lambda:)$ by $\varphi \cdot i_{2\#}$. Let Y_1 be a path-component of $\tilde{Y}$. Then, by Lemma 2.1, $p|_{Y_1}: Y_1 \rightarrow Y$ is an infinite cyclic covering space of Y belonging to $\text{Ker } (\varphi \cdot i_{2\#})$. Y is the complement of an open tube of K in

S^{n+2} and $\varphi \cdot i_{2\#}$ is the homomorphism given by (2.1) if we replace t by t^λ in (2.1). From these observations it follows that

$$\text{order}_{\Gamma^\lambda} H_q(Y_1; Q) = \Delta_K^q(t^\lambda).$$

Hence, by Corollary 2.2, we have

$$(3.3) \quad \text{order}_\Gamma H_q(\tilde{Y}; Q) = \Delta_K^q(t^\lambda).$$

The homomorphism

$$\pi_1(W, x) \xrightarrow{i_{1\#}} \pi_1(X, x) \xrightarrow{\varphi} G$$

is given by

$$(\varphi \cdot i_{1\#})(g) = t^{\text{Link}(g, L)} \quad \text{for } g \in \pi_1(W, x),$$

and is onto. Hence, by Lemma 2.1, $p| \tilde{W}: \tilde{W} \rightarrow W$ is an infinite cyclic covering space belonging to $\text{Ker}(\varphi \cdot i_{1\#})$.

$\pi_1(T, x)$ is mapped onto H by $\varphi \cdot i_{3\#}$. If T_1 is a path-component of $\tilde{T}$, then T_1 is homeomorphic to $S^n \times R^1$ and

$$\begin{aligned} \text{order}_{\Gamma^\lambda} H_q(T_1; Q) &= t^\lambda - 1, \quad q = 0, n, \\ &= 1, \quad \text{otherwise.} \end{aligned}$$

Hence, by Corollary 2.2, we have

$$(3.4) \quad \begin{aligned} \text{order}_\Gamma H_q(\tilde{T}; Q) &= t^\lambda - 1, \quad q = 0, n, \\ &= 1, \quad \text{otherwise.} \end{aligned}$$

By using the Mayer-Vietoris sequence of $\tilde{W}$ and $\tilde{Y}$, (3.4) and Proposition 2.5(3), we can show that

$$H_q(\tilde{X}; Q) \cong_\Gamma H_q(\tilde{W}; Q) \oplus H_q(\tilde{Y}; Q) \quad \text{for } 1 \leq q \leq n-1,$$

and

$$0 \rightarrow H_n(\tilde{T}; Q) \rightarrow H_n(\tilde{W}; Q) \oplus H_n(\tilde{Y}; Q) \rightarrow H_n(\tilde{X}; Q) \rightarrow 0$$

is exact.

By (3.3), it is clear that

$$\Delta_L^q(t) = \Delta_K^q(t^\lambda) \cdot \text{order}_\Gamma H_q(\tilde{W}; Q) \quad \text{for } 1 \leq q \leq n-1.$$

From (3.3), (3.4) and Assertion 1 of [3] it follows that

$$(t^\lambda - 1) \cdot \Delta_L^n(t) = \Delta_K^n(t^\lambda) \cdot \text{order}_\Gamma H_n(\tilde{W}; Q).$$

Hence we have proved (3.1).

We now consider the case $\lambda=0$. By (3.2), the homomorphism

$$\pi_1(Y, x) \xrightarrow{i_{2\#}} \pi_1(X, x) \xrightarrow{\varphi} G$$

is trivial. Since Y has the homology of a circle, it follows from Corollary 2.3 that

$$\begin{aligned} H_q(\tilde{Y}; Q) &\cong \Gamma, \quad q = 0, 1, \\ &\cong 0, \quad \text{otherwise.} \end{aligned}$$

Similarly

$$\pi_1(T, x) \xrightarrow{i_{3\#}} \pi_1(X, x) \xrightarrow{\varphi} G$$

is trivial and $H_q(\tilde{T}; Q) \cong 0$ for $q \neq 0, 1, n, n+1$. As before, $p|_{\tilde{W}}: \tilde{W} \rightarrow W$ is an infinite cyclic covering space belonging to $\text{Ker}(\varphi \cdot i_{1\#})$. Hence the Mayer-Vietoris sequence of $\tilde{W}$ and $\tilde{Y}$ implies that

$$(3.5) \quad 0 \rightarrow H_1(\tilde{T}; Q) \rightarrow H_1(\tilde{W}; Q) \oplus H_1(\tilde{Y}; Q) \rightarrow H_1(\tilde{X}; Q) \rightarrow 0$$

is exact,

$$(3.6) \quad H_q(\tilde{X}; Q) \cong H_q(\tilde{W}; Q) \quad \text{for } q \neq 1, n,$$

and if $n \geq 2$,

$$(3.7) \quad 0 \rightarrow H_n(\tilde{T}; Q) \rightarrow H_n(\tilde{W}; Q) \rightarrow H_n(\tilde{X}; Q) \rightarrow 0$$

is exact.

If $n \geq 2$, $H_1(T) \rightarrow H_1(Y)$ is an isomorphism, and hence $H_1(\tilde{T}; Q) \rightarrow H_1(\tilde{Y}; Q)$ is a Γ -isomorphism. By using this fact and (3.5), one can show easily that

$$H_1(\tilde{W}; Q) \cong_{\Gamma} H_1(\tilde{X}; Q).$$

Therefore, if $n \geq 2$,

$$(3.8) \quad \Delta_q^L(t) = \text{order}_{\Gamma} H_q(\tilde{W}; Q), \quad 1 \leq q \leq n-1,$$

and

$$(3.9) \quad 0 \rightarrow H_n(\tilde{T}; Q) \rightarrow H_n(\tilde{W}; Q) \rightarrow H_n(\tilde{X}; Q) \rightarrow 0$$

is exact.

If $n = 1$, let R and S be a meridian and a longitude of V respectively. R represents a generator of $H_1(Y)$ and S is a trivial element in $H_1(Y)$. Put $\tilde{R} = p^{-1}(R)$ and $\tilde{S} = p^{-1}(S)$. Then it is easy to show that

$$\begin{aligned} H_1(\tilde{T}) &\cong H_1(\tilde{R}) \oplus H_1(\tilde{S}), \\ H_1(\tilde{R}) &\rightarrow H_1(\tilde{Y}) \quad \text{is an isomorphism, and} \\ H_1(\tilde{S}) &\rightarrow H_1(\tilde{Y}) \quad \text{is trivial.} \end{aligned}$$

From this fact and (3.5) it follows that

$$(3.10) \quad 0 \rightarrow H_1(\tilde{S}; Q) \rightarrow H_1(\tilde{W}; Q) \rightarrow H_1(\tilde{X}; Q) \rightarrow 0$$

is exact.

Now let K' , V' , L' and λ be as in Theorem 1.1. The previous argument is also valid for this case. We will put a superscript “'” to the notations in the previous

argument to denote the corresponding objects for the case K' , V' , L' and λ . We may assume that $f(U')=U$ and $f(x')=x$.

It is easy to show that $f_{\#}: \pi_1(W', x') \rightarrow \pi_1(W, x)$ is an isomorphism which makes the diagram

$$\begin{array}{ccc} \pi_1(W', x') & \xrightarrow{\varphi'} & G \\ \downarrow f_{\#} & \nearrow \varphi & \\ \pi_1(W, x) & & \end{array}$$

commutative. Hence f induces an orientation-preserving homeomorphism

$$\tilde{f}: \tilde{W}' \rightarrow \tilde{W}$$

which is compatible with t and

$$(3.11) \quad \tilde{f}_{\#}: H_q(\tilde{W}'; Q) \cong_{\Gamma} H_q(\tilde{W}; Q)$$

for every integer q .

Since K' is trivial, if $\lambda \neq 0$, we have

$$\begin{aligned} \Delta_L^q(t) &= \text{order}_{\Gamma} H_q(\tilde{W}'; Q), \quad 1 \leq q \leq n-1, \\ (t^{\lambda}-1) \cdot \Delta_L^n(t) &= \text{order}_{\Gamma} H_n(\tilde{W}'; Q). \end{aligned}$$

Hence, if $\lambda \neq 0$, Theorem 1.1 follows from this fact, (3.1) and (3.11).

If $\lambda=0$ and $n \geq 2$, by (3.8) and (3.11) we have $\Delta_L^q(t) = \Delta_L^q(t)$ for $1 \leq q \leq n-1$. Moreover, the diagram

$$\begin{array}{ccc} H_n(\tilde{T}'; Q) & \longrightarrow & H_n(\tilde{W}'; Q) \\ \tilde{f}_{\#} \downarrow \cong_{\Gamma} & & \cong_{\Gamma} \downarrow \tilde{f}_{\#} \\ H_n(\tilde{T}; Q) & \longrightarrow & H_n(\tilde{W}; Q) \end{array}$$

is commutative. Hence, by (3.9), there exists a Γ -isomorphism $H_n(\tilde{X}'; Q) \cong H_n(\tilde{X}; Q)$, which implies $\Delta_L^n(t) = \Delta_L^n(t)$.

If $\lambda=0$ and $n=1$, we may assume that $f(S')=S$. The diagram

$$\begin{array}{ccc} H_1(\tilde{S}'; Q) & \longrightarrow & H_1(\tilde{W}'; Q) \\ \tilde{f}_{\#} \downarrow \cong_{\Gamma} & & \tilde{f}_{\#} \downarrow \cong_{\Gamma} \\ H_1(\tilde{S}; Q) & \longrightarrow & H_1(\tilde{W}; Q) \end{array}$$

is commutative. Hence, by (3.10), we have $H_1(\tilde{X}'; Q) \cong_{\Gamma} H_1(\tilde{X}; Q)$, which yields $\Delta_L^1(t) = \Delta_L^1(t)$.

This completes the proof of Theorem 1.1.

REMARK. With slight modifications we can show that if $\lambda=0$, then $H_q(\tilde{X}') \cong_{\Lambda} H_q(\tilde{X})$ for every integer q .

4. Proof of Theorem 1.2. Suppose that $n=2m-1$. In the case $m=1$, Theorem 1.2 follows from Theorem 9 of [5] and the equivalence of Erle's definition and Trotter's [1]. Hence we will assume that $m \geq 2$. We will use the same notations as in §3.

Let K, V, L and λ be as in Theorem 1.2. The Mayer-Vietoris sequence of $(\tilde{W}, \partial\tilde{W})$ and $(\tilde{Y}, \partial\tilde{Y})$ yields

$$(4.1) \quad \begin{aligned} H_q(\tilde{X}, \partial\tilde{W}) &\xleftarrow{\cong} H_q(\tilde{W}, \partial\tilde{W}) \oplus H_q(\tilde{Y}, \partial\tilde{Y}) \quad \text{and} \\ H^q(\tilde{X}, \partial\tilde{W}) &\xrightarrow{\cong} H^q(\tilde{W}, \partial\tilde{W}) \oplus H^q(\tilde{Y}, \partial\tilde{Y}) \end{aligned}$$

for every integer q .

As in §3, we will divide our consideration into two cases. First we will consider the case $\lambda \neq 0$. By Lemma 2.1, $\tilde{Y}$ is the disjoint union of $Y_1, \dots, Y_\mu$ such that

- (1) $\mu = |\lambda|$,
- (2) $p|Y_i: Y_i \rightarrow Y$ is an infinite cyclic covering space of Y belonging to

$$\text{Ker}(\varphi \cdot i_{2\#}),$$

where $\varphi \cdot i_{2\#}$ is given by (3.2),

- (3) t carries Y_1 to Y_2 , Y_2 to $Y_3, \dots, Y_\mu$ to Y_1 .

Note that Y is the complement of an open tube of K in S^{2m+1} .

We denote the inclusion maps as follows:

$$\begin{aligned} j: (\tilde{X}, \partial\tilde{X}) &\subset (\tilde{X}, \partial\tilde{W}), \\ j_0: (\tilde{W}, \partial\tilde{W}) &\subset (\tilde{X}, \partial\tilde{W}), \\ j_\nu: (Y_\nu, \partial Y_\nu) &\subset (\tilde{X}, \partial\tilde{W}), \quad \nu = 1, 2, \dots, \mu. \end{aligned}$$

Let $e \in H_{2m-1}(\partial\tilde{X})$ be the canonical generator and $\zeta = \partial_{\star}^{-1}(e)$ the fundamental class of $\tilde{X}$. For $\nu = 1, \dots, \mu$, let $e_\nu \in H_{2m-1}(\partial Y_\nu)$ be the canonical generator and $\zeta_\nu = \partial_{\star}^{-1}(e_\nu)$ the fundamental class of Y_ν , where

$$\partial_{\star}: H_{2m}(Y_\nu, \partial Y_\nu) \xrightarrow{\cong} H_{2m-1}(\partial Y_\nu).$$

Since $t_\star(e_1) = e_2, \dots, t_\star(e_\mu) = e_1$, we have

$$(4.2) \quad t_\star(\zeta_1) = \zeta_2, \quad t_\star(\zeta_2) = \zeta_3, \quad \dots, \quad t_\star(\zeta_\mu) = \zeta_1.$$

By Proposition 2.5(3), $H_{2m}(\tilde{X}) \cong 0$ and $H_{2m}(Y_\nu) \cong 0$ for $\nu = 1, 2, \dots, \mu$. Hence the Mayer-Vietoris sequence of $\tilde{W}$ and $\tilde{Y}$ yields $H_{2m}(\tilde{W}) \cong 0$, which implies that $\partial_{0\star}: H_{2m}(\tilde{W}, \partial\tilde{W}) \rightarrow H_{2m-1}(\partial\tilde{W})$ is one-one.

We want to show that

(4.3) there exists a unique element $\zeta_0 \in H_{2m}(\tilde{W}, \partial\tilde{W})$ such that

$$\partial_{0\star}(\zeta_0) = e - (\text{sign } \lambda) \cdot (e_1 + \dots + e_\mu).$$

The isomorphism

$$h_*: H_{2m}(\tilde{W}, \partial \tilde{W}) \oplus H_{2m}(Y_1, \partial Y_1) \oplus \cdots \oplus H_{2m}(Y_\mu, \partial Y_\mu) \rightarrow H_{2m}(\tilde{X}, \partial \tilde{W})$$

is given by

$$(4.4) \quad h_*(x_0, x_1, \dots, x_\mu) = \sum_{v=0}^{\mu} j_{v*}(x_v)$$

for $x_0 \in H_{2m}(\tilde{W}, \partial \tilde{W})$ and $x_v \in H_{2m}(Y_v, \partial Y_v)$, $v=1, 2, \dots, \mu$. There exist a unique element $\zeta_0 \in H_{2m}(\tilde{W}, \partial \tilde{W})$ and unique integers $\varepsilon_1, \dots, \varepsilon_\mu$ such that

$$(4.5) \quad j_*(\zeta) = h_*(\zeta_0, \varepsilon_1 \zeta_1, \dots, \varepsilon_\mu \zeta_\mu).$$

If we operate t_* on (4.5), by (4.2) and the fact $t_*(\zeta) = \zeta$, we obtain

$$j_*(\zeta) = h_*(t_*(\zeta_0), \varepsilon_\mu \zeta_1, \varepsilon_1 \zeta_2, \dots, \varepsilon_{\mu-1} \zeta_\mu).$$

Since h_* is one-one, we have $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_\mu$.

By the commutativity of the following diagrams

$$\begin{array}{ccc} H_{2m}(\tilde{X}, \partial \tilde{X}) & \xrightarrow{j_*} & H_{2m}(\tilde{X}, \partial \tilde{W}) \\ \downarrow \partial_* & & \downarrow \bar{\partial}_* \\ H_{2m-1}(\partial \tilde{X}) & \longrightarrow & H_{2m-1}(\partial \tilde{W}) \end{array}, \quad \begin{array}{ccc} H_{2m}(\tilde{W}, \partial \tilde{W}) & \xrightarrow{j_{0*}} & H_{2m}(\tilde{X}, \partial \tilde{W}) \\ \downarrow \partial_{0*} & & \downarrow \bar{\partial}_* \\ H_{2m-1}(\partial \tilde{W}) & \longrightarrow & H_{2m-1}(\partial \tilde{W}) \end{array}$$

and

$$\begin{array}{ccc} H_{2m}(W_v, \partial W_v) & \xrightarrow{j_{v*}} & H_{2m}(\tilde{X}, \partial \tilde{W}) \\ \downarrow \partial_{v*} & & \downarrow \bar{\partial}_* \\ H_{2m-1}(\partial W_v) & \longrightarrow & H_{2m-1}(\partial \tilde{W}) \end{array}$$

we obtain

$$(4.6) \quad \bar{\partial}_* j_*(\zeta) = e, \quad \bar{\partial}_* h_{0*}(\zeta_0) = \partial_{0*}(\zeta_0), \quad \text{and} \quad \bar{\partial}_* j_{v*}(\zeta_v) = e_v, \\ v = 1, 2, \dots, \mu.$$

Hence, by (4.4)–(4.6), we have $\partial_{0*}(\zeta_0) = e - \varepsilon_1(e_1 + \cdots + e_\mu)$ in $H_{2m-1}(\partial \tilde{W})$.

To complete (4.3), we need only to show that $\varepsilon_1 = \text{sign } \lambda$. Since the diagram

$$\begin{array}{ccc} H_{2m}(\tilde{W}, \partial \tilde{W}) & \xrightarrow{p_*} & H_{2m}(W, \partial W) \\ \downarrow \partial_{0*} & & \downarrow \bar{\partial}_{0*} \\ H_{2m-1}(\partial \tilde{W}) & \xrightarrow{p_*} & H_{2m-1}(\partial W) \end{array}$$

is commutative, in $H_{2m-1}(\partial W)$ we have

$$\bar{\partial}_{0*}(p_*(\zeta_0)) = p_*(\partial_{0*}(\zeta_0)) = p_*(e) - \varepsilon_1(p_*(e_1) + \cdots + p_*(e_\mu)).$$

Hence $p_*(e) = \varepsilon_1(p_*(e_1) + \cdots + p_*(e_\mu))$ in $H_{2m-1}(W)$, and in $H_{2m-1}(V)$. This shows that $L \sim \varepsilon_1 \mu K$ in V and $\varepsilon_1 = \text{sign } \lambda$. This completes the proof of (4.3).

Since $m \geq 2$, $j^*: H^m(\tilde{X}, \partial \tilde{W}) \rightarrow H^m(\tilde{X}, \partial \tilde{X})$ is an isomorphism. The homomorphism

$$h^*: H^m(X, \partial \tilde{W}) \rightarrow H^m(\tilde{W}, \partial \tilde{W}) \oplus H^m(Y_1, \partial Y_1) \oplus \cdots \oplus H^m(Y_\mu, \partial Y_\mu)$$

given by

$$h^*(x) = (j_0^*(x), j_1^*(x), \dots, j_\mu^*(x))$$

for $x \in H^m(\tilde{X}, \partial \tilde{W})$ is an isomorphism. For an element x of $H^m(\tilde{X}, \partial \tilde{X})$, we denote by $\bar{x}$ the image of x under the isomorphism $h^* \cdot j^{*-1}$.

Let $x_1, \dots, x_r$ and $y_1, \dots, y_s$ be elements of $H^m(\tilde{X}, \partial \tilde{X}; Q)$ such that

$\{\bar{x}_1, \dots, \bar{x}_r\}$ is a basis for $H^m(\tilde{W}, \partial \tilde{W}; Q)$ and

$\{\bar{y}_1, \dots, \bar{y}_s\}$ is a basis for $H^m(Y_\mu, \partial Y_\mu; Q)$.

Then $\{t^{*\nu} \bar{y}_1, \dots, t^{*\nu} \bar{y}_s\}$ is a basis for $H^m(W_{\mu-\nu}, \partial W_{\mu-\nu}; Q)$ for $\nu=0, 1, \dots, \mu-1$, and

$$\mathcal{B}: x_1, \dots, x_r; y_1, \dots, y_s; \dots; t^{*\mu-1} y_1, \dots, t^{*\mu-1} y_s$$

forms a basis for $H^m(\tilde{X}, \partial \tilde{X}; Q)$.

First we want to consider the matrix A representing the pairing B_1 with respect to the basis $\mathcal{B}$. By making use of the fact that $j_*(\zeta) = h_*(\zeta_0, \varepsilon \zeta_1, \dots, \varepsilon \zeta_\mu)$, where $\varepsilon = \text{sign } \lambda$, we can show that

$$\begin{aligned} B_1(x_i, x_j) &= \langle \bar{x}_i \cup \bar{x}_j, \zeta_0 \rangle, \\ B_1(t^{*\nu} y_i, t^{*\nu} y_j) &= \varepsilon \langle \bar{y}_i \cup \bar{y}_j, \zeta_\mu \rangle, & 0 \leq \nu \leq \mu-1, \\ B_1(x_i, t^{*\nu} y_j) &= B_1(t^{*\nu} y_i, x_j) = 0, & 0 \leq \nu \leq \mu-1, \\ B_1(t^{*\nu} y_i, t^{*\tau} y_j) &= 0, & \nu \neq \tau. \end{aligned}$$

Let

$$C = \|\langle \bar{x}_i \cup \bar{x}_j, \zeta_0 \rangle\|_{1 \leq i, j \leq r}$$

and

$$D = \|\varepsilon \langle \bar{y}_i \cup \bar{y}_j, \zeta_\mu \rangle\|_{1 \leq i, j \leq s}.$$

Then we have

$$A = \begin{pmatrix} C & & 0 \\ & D & \\ & & \ddots \\ 0 & & & D \end{pmatrix}.$$

If m is even, then

$$\text{signature } \|\langle \bar{y}_i \cup \bar{y}_j, \zeta_\mu \rangle\|_{1 \leq i, j \leq s} = \sigma_1(K)$$

and

$$(4.7) \quad \sigma_1(L) = \text{signature } C + \varepsilon \mu \sigma_1(K) = \text{signature } C + \lambda \sigma_1(K).$$

We now want to consider the matrix B representing B_2 with respect to $\mathcal{B}$. We can show that

$$\begin{aligned}
 B_2(x_i, x_j) &= \langle \bar{x}_i \cup t^* \bar{x}_j + \bar{x}_j \cup t^* \bar{x}_i, \zeta_0 \rangle, \\
 B_2(t^{*\nu} y_i, t^{*\nu+1} y_j) &= \varepsilon \langle \bar{y}_j \cup \bar{y}_i, \zeta_\mu \rangle, \\
 B_2(y_i, t^{*\mu-1} y_j) &= \varepsilon \langle \bar{y}_i \cup t^* \bar{y}_j + \bar{y}_j \cup t^* \bar{y}_i, \zeta_1 \rangle \quad \text{if } \mu = 1, \\
 &= \varepsilon \langle y_i \cup t^{*\mu} y_j, \zeta_\mu \rangle \quad \text{if } \mu > 1, \\
 B_2(t^{*\nu} y_i, t^{*\tau} y_j) &= 0 \quad \text{if } \mu > 1 \text{ and } |\nu - \tau| \neq 1, \mu - 1, \\
 B_2(x_i, t^{*\nu} y_j) &= 0.
 \end{aligned}$$

Let

$$E = \|\langle \bar{x}_i \cup t^* \bar{x}_j + \bar{x}_j \cup t^* \bar{x}_i, \zeta_0 \rangle\|_{1 \leq i, j \leq r}$$

and

$$F = \|\varepsilon \langle \bar{y}_i \cup t^{*\mu} \bar{y}_j, \zeta_\mu \rangle\|_{1 \leq i, j \leq s}.$$

Then we have

$$B = \begin{array}{c} \mu \text{ blocks} \end{array} \left[\begin{array}{c|c} E & \overbrace{\begin{array}{cccc} 0 & & & \\ 0 & D' & & F \\ D & 0 & D' & \\ & D & 0 & 0 \\ & & \ddots & \ddots \\ 0 & & & \ddots & \ddots \\ & 0 & & D & 0 & D' \\ F' & & & D & 0 \end{array}}^{\mu \text{ blocks}} \end{array} \right]$$

If μ is even, it is easy to show that the matrix

$$(4.8) \quad \begin{array}{c} \mu \end{array} \left[\begin{array}{cccc} 0 & D' & & F \\ D & 0 & D' & \\ & D & 0 & 0 \\ & & \ddots & \ddots \\ 0 & & & \ddots & \ddots \\ & 0 & & D & 0 & D' \\ F' & & & D & 0 \end{array} \right]$$

is congruent to its negative and its signature is zero. Hence if μ is even,

$$(4.9) \quad \sigma_2(L) = \text{signature } E.$$

Proposition 2.6 implies that $D' = (-1)^m D$ and D is nonsingular. If μ is odd, by adding $(-1)^{m+1}$ times the first row block to the third and $(-1)^{m+1}$ times the first column block to the third; $(-1)^{m+1}$ times the new third row block to the 5th and $(-1)^{m+1}$ times the new third column block to the 5th; $\dots$; $(-1)^{m+1}$ times the new $(\mu-2)$ th row block to the μ th and $(-1)^{m+1}$ times the new $(\mu-2)$ th column block to the μ th, we can show that (4.8) is congruent to the matrix

$$(4.10) \quad \begin{bmatrix} \boxed{\begin{smallmatrix} 0 & D' \\ D & 0 \end{smallmatrix}} & & & F & & \\ & & & 0 & & 0 \\ & \boxed{\begin{smallmatrix} 0 & D' \\ D & 0 \end{smallmatrix}} & & F_1 & & \\ & & & 0 & & \\ & & \ddots & & & \\ & \frac{\mu-1}{2} & & & & \\ & & & \boxed{\begin{smallmatrix} 0 & D' \\ D & 0 \end{smallmatrix}} & F_{(\mu-3)/2} & \\ & 0 & & & 0 & \\ F' & 0 & F'_1 & 0 & \dots & F'_{(\mu-3)/2} & 0 & F_{(\mu-1)/2} + F'_{(\mu-1)/2} \end{bmatrix}$$

where $F_i = (-1)^{(m+1)i} F$, $i = 1, \dots, (\mu-1)/2$. Since D is nonsingular, (4.10) is congruent to

$$\begin{bmatrix} \boxed{\begin{smallmatrix} 0 & D' \\ D & 0 \end{smallmatrix}} & & & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & \boxed{\begin{smallmatrix} 0 & D' \\ D & 0 \end{smallmatrix}} & & \\ 0 & & & & & F_{(\mu-1)/2} + F'_{(\mu-1)/2} \end{bmatrix}$$

It is clear that signature $\begin{bmatrix} 0 & D' \\ D & 0 \end{bmatrix} = 0$. One can show that

$$\begin{aligned} F_{(\mu-1)/2} + F'_{(\mu-1)/2} &= (-1)^{(m+1) \cdot (\mu-1)/2} \|e \langle \bar{y}_i \cup t^{*\mu} \bar{y}_j + \bar{y}_j \cup t^{*\mu} \bar{y}_i, \zeta_\mu \rangle\|_{1 \leq i, j \leq s} \\ &= (-1)^{(m+1) \cdot (\lambda-1)/2} \|\langle \bar{y}_i \cup t^{*\lambda} \bar{y}_j + \bar{y}_j \cup t^{*\lambda} \bar{y}_i, \zeta_\mu \rangle\|_{1 \leq i, j \leq s}. \end{aligned}$$

Therefore, if μ is odd,

$$(4.11) \quad \sigma_2(L) = \text{signature } E + (-1)^{(m+1) \cdot (\lambda-1)/2} \sigma_2(K).$$

Now, as in §3, $\tilde{f}^*: H^m(\tilde{W}, \partial \tilde{W}) \rightarrow H^m(\tilde{W}', \partial \tilde{W}')$ is an isomorphism which is compatible with t^* and $\tilde{f}_*: H_{2m}(\tilde{W}', \partial \tilde{W}') \rightarrow H_{2m}(\tilde{W}, \partial \tilde{W})$ is an isomorphism such

that the diagram

$$\begin{array}{ccc} H_{2m}(\tilde{W}', \partial \tilde{W}') & \xrightarrow[\cong]{\tilde{f}_*} & H_{2m}(\tilde{W}, \partial \tilde{W}) \\ \downarrow \partial'_{0*} & & \downarrow \partial_{0*} \\ H_{2m-1}(\partial \tilde{W}') & \xrightarrow[\cong]{\tilde{f}_*} & H_{2m-1}(\partial \tilde{W}) \end{array}$$

is commutative. Since $\tilde{f}_*(e') = e$ and $\tilde{f}_*(e'_1 + \cdots + e'_\mu) = e_1 + \cdots + e_\mu$,

$$\begin{aligned} \partial_{0*}(\tilde{f}_*(\zeta'_0)) &= \tilde{f}_*(\partial'_{0*}(\zeta'_0)) = \tilde{f}_*(e' - (\text{sign } \lambda) \cdot (e'_1 + \cdots + e'_\mu)) \\ &= e - (\text{sign } \lambda) \cdot (e_1 + \cdots + e_\mu). \end{aligned}$$

Hence, by (4.3), we have $\tilde{f}_*(\zeta'_0) = \zeta_0$.

$\{\tilde{f}^*(\bar{x}_1), \dots, \tilde{f}^*(\bar{x}_r)\}$ is a basis for $H^m(\tilde{W}', \partial \tilde{W}'; \mathcal{Q})$ which satisfies

$$\langle \tilde{f}^*(\bar{x}_i) \cup \tilde{f}^*(\bar{x}_j), \zeta'_0 \rangle = \langle \bar{x}_i \cup \bar{x}_j, \zeta_0 \rangle$$

and

$$\langle \tilde{f}^*(\bar{x}_i) \cup t^* \tilde{f}^*(\bar{x}_j) + \tilde{f}^*(\bar{x}_j) \cup t^* \tilde{f}^*(\bar{x}_i), \zeta'_0 \rangle = \langle \bar{x}_i \cup t^* \bar{x}_j + \bar{x}_j \cup t^* \bar{x}_i, \zeta_0 \rangle.$$

Since K' is trivial, by (4.7), (4.9) and (4.11), we have

$$\sigma_1(L') = \text{signature } \|\langle \tilde{f}^*(\bar{x}_i) \cup \tilde{f}^*(\bar{x}_j), \zeta'_0 \rangle\| = \text{signature } C$$

and

$$\sigma_2(L') = \text{signature } \|\langle \tilde{f}^*(\bar{x}_i) \cup t^* \tilde{f}^*(\bar{x}_j) + \tilde{f}^*(\bar{x}_j) \cup t^* \tilde{f}^*(\bar{x}_i), \zeta'_0 \rangle\| = \text{signature } E.$$

Therefore we have shown that

$$\begin{aligned} \sigma_2(L) &= \sigma_2(L') && \text{when } \lambda \text{ is even,} \\ &= \sigma_2(L') + (-1)^{(m+1) \cdot (\lambda-1)/2} \sigma_2(K) && \text{when } \lambda \text{ is odd,} \end{aligned}$$

and if m is even,

$$\sigma_1(L) = \sigma_1(L') + \lambda \sigma_1(K).$$

This completes the case $\lambda \neq 0$.

Finally we want to consider the case $\lambda = 0$. $\tilde{Y}$ consists of countably many copies of Y and $\partial \tilde{Y}$ consists of countably many copies of $\partial Y \cong S^{2m-1} \times S^1$. In the Mayer-Vietoris sequence of $(\tilde{Y}, \emptyset)$ and $(\tilde{W}, \partial \tilde{X})$, we can show that

$$H^m(\tilde{X}, \partial \tilde{X}) \xrightarrow{h^*} H^m(\tilde{W}, \partial \tilde{X})$$

is an isomorphism and

$$0 \longrightarrow H_{2m}(\partial \tilde{Y}) \longrightarrow H_{2m}(\tilde{W}, \partial \tilde{X}) \xrightarrow{h_*} H_{2m}(\tilde{X}, \partial \tilde{X}) \longrightarrow 0$$

is exact. Note that $m \geq 2$. Hence there exists an element $\zeta_0 \in H_{2m}(\tilde{W}, \partial \tilde{X})$ such that $h_*(\zeta_0) = \zeta$.

Let $\{x_1, \dots, x_r\}$ be a basis for $H^m(\tilde{X}, \partial\tilde{X}; Q)$. Then $\{\bar{x}_1, \dots, \bar{x}_r\}$ is a basis for $H^m(\tilde{W}, \partial\tilde{X}; Q)$, where $\bar{x}_i = h^*(x_i)$ for $i=1, \dots, r$, and

$$B_1(x_i, x_j) = \langle \bar{x}_i \cup \bar{x}_j, \zeta_0 \rangle,$$

$$B_2(x_i, x_j) = \langle \bar{x}_i \cup t^* \bar{x}_j + \bar{x}_j \cup t^* \bar{x}_i, \zeta_0 \rangle.$$

As before, $\tilde{f}^*: H^m(\tilde{W}, \partial\tilde{X}) \rightarrow H^m(\tilde{W}', \partial\tilde{X}')$ and $\tilde{f}_*: H_{2m}(\tilde{W}', \partial\tilde{X}') \rightarrow H_{2m}(\tilde{W}, \partial\tilde{X})$ are isomorphisms and $\tilde{f}_*(e') = e$. Since the diagram

$$\begin{array}{ccccc} & H_{2m}(\tilde{W}', \partial\tilde{X}') & \xrightarrow[\cong]{\tilde{f}_*} & H_{2m}(\tilde{W}, \partial\tilde{X}) & \\ & \downarrow \partial'_0 * & & \downarrow \partial_0 * & \\ h_* \swarrow & & & & \searrow h_* \\ H_{2m}(\tilde{X}', \partial\tilde{X}') & \xrightarrow[\cong]{\partial'_*} & H_{2m-1}(\partial\tilde{X}') & \xrightarrow[\cong]{\tilde{f}_*} & H_{2m-1}(\partial\tilde{X}) \xleftarrow[\cong]{\partial_*} H_{2m}(\tilde{X}, \partial\tilde{X}) \end{array}$$

is commutative, $\zeta'_0 = \tilde{f}_*^{-1}(\zeta_0) \in H_{2m}(\tilde{W}', \partial\tilde{X}')$ satisfies $h'_*(\zeta'_0) = \zeta_0$.

$\{\tilde{f}^*(\bar{x}_1), \dots, \tilde{f}^*(\bar{x}_r)\}$ is a basis for $H^m(\tilde{W}', \partial\tilde{X}'; Q)$ and $\{x'_1, \dots, x'_r\}$ is a basis for $H^m(\tilde{X}', \partial\tilde{X}'; Q)$, where $x'_i = h'^{-1}(\tilde{f}^*(\bar{x}_i))$ for $i=1, \dots, r$. Hence

$$\begin{aligned} B_1(x'_i, x'_j) &= \langle \tilde{f}^*(\bar{x}_i) \cup \tilde{f}^*(\bar{x}_j), \zeta'_0 \rangle \\ &= \langle \bar{x}_i \cup \bar{x}_j, \zeta_0 \rangle = B_1(x_i, x_j). \end{aligned}$$

Likewise we have $B_2(x'_i, x'_j) = B_2(x_i, x_j)$. Therefore we have shown that $\sigma_2(L') = \sigma_2(L)$ and, if m is even, $\sigma_1(L') = \sigma_1(L)$. This completes the proof of Theorem 1.2.

REFERENCES

1. D. Erle, *Die quadratische Form eines Knotens und ein Satz über Knotenmannigfaltigkeiten*, J. Reine Angew. Math. **236** (1969), 174–218. MR **40** #2106.
2. ———, *Quadratische Formen als Invarianten von Einbettungen der Kodimension 2*, Topology **8** (1969), 99–114. MR **38** #6576.
3. J. W. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds (Michigan State University, E. Lansing, Mich., 1967), Prindle, Weber and Schmidt, Boston, Mass., 1968, pp. 115–133. MR **39** #3497.
4. H. Seifert, *On the homology invariants of knots*, Quart J. Math. Oxford Ser. (2) **1** (1950), 23–32. MR **11**, 735.
5. Y. Shinohara, *On the signature of knots and links*, Trans. Amer. Math. Soc. **156** (1971), 273–285.
6. H. F. Trotter, *Homology of group systems with applications to knot theory*, Ann. of Math. (2) **76** (1962), 464–498. MR **26** #761.

UNIVERSITY OF TORONTO, TORONTO, CANADA